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Quantum Phase Transition Like Phenomenon in a Two-Qubit Yang-Baxter System

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Abstract The quantum phase transition for the "*q-deformed*" Yang-baxter Hamiltonian has been discussed. The calculation shows when the deformed parameter q approaches 1, there exists a quantum critical point for spectral parameter θ. In this Yang-Baxter system, quantum entanglement and the geometric phase can characterize quantum phase transition.

Keywords Entanglement · Berry phase · Yang-Baxter system

1 Introduction

A quantum phase transition (QPT), a dramatic change in the ground state driven by parameters at zero temperature, is associated with a level crossing (LC) or avoided level crossing (ALC) between the ground and the excited energy level [1]. Such phase transition being purely quantum phenomenon is driven by quantum fluctuations. Energy is the most primary quantity for determining the QPT phenomenon. In Refs. [2, 3], the authors point that quantum phase transitions are caused by the reconstruction of the Hamiltonian's energy spectra, especially of the low-lying excitation spectra.

Recently, the intriguing issue of the relation between quantum entanglement (QE), geometric phase (GP) [4, 5] and QPT has been emerged, attracting much attention [6–16]. Since QPTs are driven by quantum fluctuations, entanglement, referring to quantum correlations between subsystems, could be a good indicator to QPTs. Indeed, in many spin models, the QPT phenomena are signaled by QE [6–11]. However, they are model dependent. Since the GP is associated with LC or ALC, the GP might have a peculiar behavior near the quantum critical point. The connection between GP and QPT has been established in many works [12–14].

Very recently, braiding operators and the Yang-Baxter Equation (YBE) [17–20], have introduced to the filed of quantum information and quantum computation processing [21–27].

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Kauffman and Lomonaco have explored the role of unitary Yang-Baxter $\breve{R}$ matrices in quantum computation [21]. It is shown that braid matrices, as well as Yang-Baxter $\breve{R}$ matrices can be identified as the universal quantum gates [24]. With the unitary $\breve{R}$ matrices, Chen et.al. constructed a set of Hamiltonians, and explored the Berry phase and quantum criticality of the Yang-Baxter system [25].

In this paper, we focus on the *q-deformed* braid group realization (BGR). With this BGR, a Yang-Baxter Hamiltonian can be constructed. In Sect. 2, we study the role of spectral parameter θ and *deformed* parameter q. In Sect. 3, we investigate the QE and the GP as indictors of QPT-Like phenomenon.

2 The Model and Its Eigen-Problem

In this paper, the matrix realizations of Temperley-Lieb Algebra (TLA) U-matrix, YBE solution $\breve{R}$-matrix and braid group realization (BGR) S-matrix are 4×4 matrices acting on the tensor product space $V \times V$, where V is a 2-dimensional vector space. As U, S and $\breve{R}$ act on the tensor product $V_i \times V_{i+1}$, we denote them by U_i, b_i and $\breve{R}_i$, respectively.

We first briefly review the theory of braid groups, the YBE and Yang-Baxterization approach [28]. Let B_n denote the braid group on n strands. B_n is generated by elementary braids $\{I, b_1, b_2, \ldots, b_{n-1}\}$ with the braid relations,

$$\begin{cases} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1}, & 1 \le i < n-2, \\ b_i b_j = b_j b_i, & |i-j| \ge 2, \end{cases} \tag{1}$$

where the notation $b_i \equiv b_{i,i+1}$ is used. The b_i represents $1_1 \otimes 1_2 \otimes 1_3 \cdots \otimes 1_{i-1} \otimes S \otimes 1_{i+2} \otimes \cdots \otimes 1_n$, and 1_j is the unit matrix of the j-th particle. Then we call S the braid group representation (BGR). In addition, the braid group is easily understood in terms of knot diagrams in Ref. [29].

As is known, a solution of YBE can be found via Yang-Baxterization acting on the solution of the braid relation. For example, if S has two eigenvalues (λ_1 , λ_2), then the Yang-Baxterization of the unitary braiding operator S is

$$\breve{R}(x) = \rho(x)(xS + x^{-1}\lambda_1\lambda_2 S^{-1}). \tag{2}$$

Let $\breve{R}_i$ denote $\breve{R}_{i,i+1}$. The unitary $\breve{R}$-matrix satisfies the YBE which is of the form,

$$\breve{R}_i(x)\breve{R}_{i+1}(xy)\breve{R}_i(y) = \breve{R}_{i+1}(y)\breve{R}_i(xy)\breve{R}_{i+1}(x), \tag{3}$$

where multiplicative parameters x and y are known as the spectral parameters. Generally, multi-spin interaction Hamiltonians can be constructed based on the YBE. As $\breve{R}$ is unitary, it can define the evolution of a state $|\Psi(0)\rangle$

$$|\Psi(t)\rangle = \breve{R}_i(t)|\Psi(0)\rangle, \tag{4}$$

here $\breve{R}_i(t)$ is time-dependent, which can be realized by specifying corresponding time-dependent parameter of $\breve{R}_i$. By taking partial derivative of the state $|\Psi(t)\rangle$ with respect to time t, we have an equation

$$\begin{aligned} i\hbar\frac{\partial|\Psi(t)\rangle}{\partial t} &= i\hbar\left[\frac{\partial \breve{R}_i(t)}{\partial t}\breve{R}_i^{\dagger}(t)\right]\breve{R}_i(t)|\Psi(0)\rangle, \\ &= H(t)|\Psi(t)\rangle, \end{aligned} \tag{5}$$

where $H(t) = i\hbar \frac{\partial \breve{R}_i(t)}{\partial t} \breve{R}_i^\dagger(t)$ is the Hamiltonian governing the evolution of the state $|\Psi(t)\rangle$. Thus, the Hamiltonian $H(t)$ for the Yang-Baxter system is derived through the Yang-Baxterization approach.

In this paper, we focus on the standard spin-1/2 six-vertex BGR [30],

$$S = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 0 & -\eta & 0 \\ 0 & -\eta^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & q \end{pmatrix} = q(I - q^{-1}U), \tag{6}$$

where U satisfies Temperley-Lieb relations, $U_i U_{i\pm 1} U_i = U_i$, $U_i U_j = U_j U_i$ (for $|i - j| \geq 2$) and $U^2 = dU$. In this case, $d = q + q^{-1}$. In topology, the parameter d corresponds to a single loop "◯". The matrix form for U is

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & \eta & 0 \\ 0 & \eta^{-1} & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{7}$$

Let the I_4 and $\mathbf{0}_4$ denote 4 dimensional identity matrix and zero matrix, respectively. We can verify that the BGR (6) has two distinct eigenvalues $\lambda_1 = q$ and $\lambda_2 = -q^{-1}$ (i.e. $(S - qI_4)(S + q^{-1}I_4) = \mathbf{0}_4$). Substituting such two eigenvalues into (2), we obtain a Yang-Baxter $\breve{R}$ matrix as follows,

$$\breve{R}(x) = [q^2 + q^{-2} - (x^2 + x^{-2})]^{-1/2}[(qx - q^{-1}x^{-1})I - (x - x^{-1})U]. \tag{8}$$

The inverse matrix of $\breve{R}(x)$ can be obtained as following,

$$[\breve{R}(x)]^{-1} = [q^2 + q^{-2} - (x^2 + x^{-2})]^{-1/2}[(qx^{-1} - q^{-1}x)I + (x - x^{-1})U]. \tag{9}$$

For our convenience, we can introduce two parameters ϑ and φ with $x = e^{i\vartheta}$ and $\eta = e^{i\varphi}$. The unitary condition $\breve{R}(\vartheta, \varphi)[\breve{R}(\vartheta, \varphi)]^\dagger = [\breve{R}(\vartheta, \varphi)]^\dagger \breve{R}(\vartheta, \varphi) = I$ gives us ϑ and φ are real. Then this Yang-Baxter Hamiltonian can be recast as follows,

$$H = \hbar\omega \sin\vartheta (Q^2 + \sin^2\vartheta)^{-1}[\sin\vartheta(S_1^3 - S_2^3) + Q(e^{-i(\varphi + \frac{\pi}{2})} S_1^+ S_2^- + e^{i(\varphi + \frac{\pi}{2})} S_1^- S_2^+)],$$

where $Q = (q - q^{-1})/2$. Let

$$v_1 = \begin{pmatrix} 0 & e^{-\frac{i\vartheta}{4}} \\ e^{\frac{i\vartheta}{4}} & 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 0 & e^{\frac{i\vartheta}{4}} \\ e^{-\frac{i\vartheta}{4}} & 0 \end{pmatrix}.$$

We can introduce a time-independent local unitary transformation as follows,

$$H' = VHV^{-1},$$

where $V = v_1 \otimes v_2$. We should note that this local unitary transformation is time independent, so the entanglement and geometric phase properties are not vary under such transformation. If we set $\vartheta = \pi/2 - \theta$ and $\varphi(t) = \phi(t) - \pi/2 = wt$, we can obtain the following Yang-Baxter Hamiltonian,

$$H' = \hbar\omega \cos\theta (Q^2 + \cos^2\theta)^{-1} H_0,$$

where

$$H_0 = \cos\theta(S_1^3 - S_2^3) + Q(e^{-i\phi}S_1^+S_2^- + e^{i\phi}S_1^-S_2^+).$$

In its matrix form,

$$H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta & Qe^{-i\phi} & 0 \\ 0 & Qe^{i\phi} & -\cos\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In fact, this Hamiltonian is defined on the subspace spanned by $\{|01\rangle, |10\rangle\}$. This looks like a Hamiltonian of a spin $1/2$ in a magnetic field. In fact, we can introduce a set of $SU(2)$ operators $\{S^{(+)}, S^{(-)}, S^{(3)}\}$, where $S^{(+)} = S_1^+S_2^-$, $S^{(-)} = S_1^-S_2^+$ and $S^{(3)} = (S_1^3 - S_2^3)/2$. In terms of this set of $SU(2)$ operators the Hamiltonian H_0 can be recast as follows,

$$H_0 = 2\sqrt{Q^2 + \cos^2\theta}\,\mathbf{B}\cdot\mathbf{S},$$

where $\mathbf{B} = (\sin\theta'\cos\phi, \sin\theta'\sin\phi, \cos\theta')$ with $\theta' = \arctan(Q/\cos\theta)$ and $\mathbf{S} = (S^{(1)}, S^{(2)}, S^{(3)})$ with $S^{(1)} = \frac{1}{2}(S^{(+)} + S^{(-)})$ and $S^{(2)} = \frac{1}{2i}(S^{(+)} - S^{(-)})$. We can easily obtain the eigenenergies of H_0 as follows,

$$E_\pm = \pm\sqrt{Q^2 + \cos^2\theta}. \tag{10}$$

Obviously, The ground state energy of this Yang-Baxter system is $E_g = E_- = -\sqrt{Q^2 + \cos^2\theta}$. And the corresponding eigenstate is

$$|\psi_g\rangle = \frac{1}{\sqrt{Q^2 + (E_g - \cos\theta)^2}}(Qe^{-i\phi}|01\rangle + (E_g - \cos\theta)|10\rangle). \tag{11}$$

Between E_+ and E_-, there exists energy gap $\Delta E = 2\sqrt{Q^2 + \cos^2\theta}$. When $Q = 0$ (i.e. $q = 1$ or single loop $d = 2$), there is a crossover point $\theta_c = \pi/2$. If $Q \neq 0$, there is a finite gap $2|Q|$ at the crossover point (please see Fig. 1(a)). We can see that when Q approaches 0 (i.e. d approaches 2), the energy gap between the ground state and excited state becomes smaller. We can obtain the first and the second derivative of the ground state energy E_g with respect to the parameter θ as follows,

$$\frac{\partial E_g}{\partial\theta} = -\frac{\sin 2\theta}{2E_g}, \qquad \frac{\partial^2 E_g}{\partial\theta^2} = -\frac{\cos 2\theta}{E_g} - \frac{\sin^2 2\theta}{4(E_g)^3}. \tag{12}$$

To see the quantum criticality obviously, we plot the first and the second derivative of the ground state energy E_g with respect to the parameter θ (Please see Figs. 1(b) and 1(c)). Obviously, when $Q \to 0$, there is quantum critical point $\theta_c = \pi/2$.

3 The QE and the GP as Witness of QPT-Like

In this section, we confine our interest at QE and GP of the ground state in (11).

To describe entanglement, we adopt the concurrence of a biparticle state to define entanglement degree of a state. For a pure two-qubit state $|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$

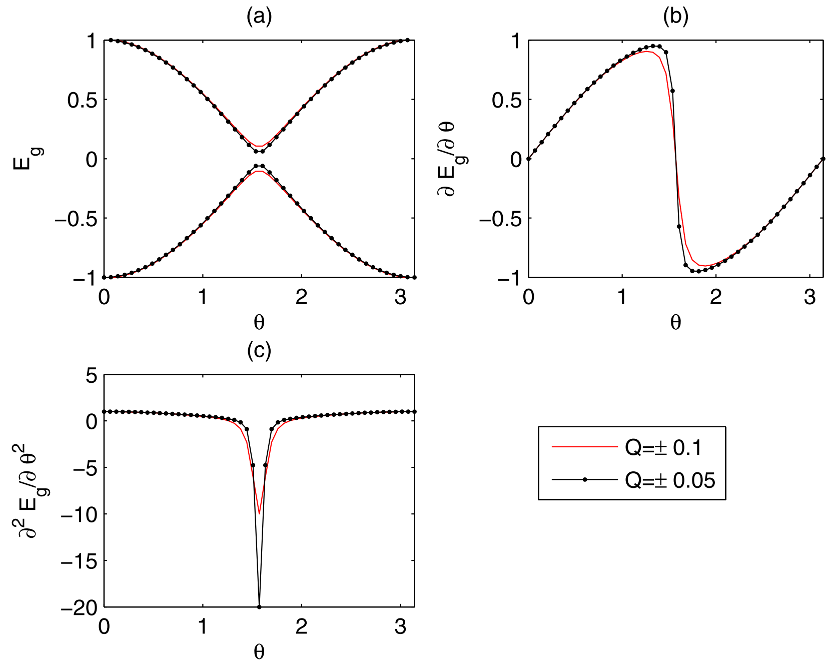

Fig. 1 (**a**) E_g as a function of θ; (**b**) $\frac{\partial E_g}{\partial\theta}$ as a function of θ; (**c**) $\frac{\partial^2 E_g}{\partial\theta^2}$ as a function of θ. The parameter $\theta \in (0,\pi)$. The *red solid line* represents $Q=\pm 0.1$ and the *black dashed line* represents $Q=\pm 0.05$

with $|a|^2+|b|^2+|c|^2+|d|^2=1$, the concurrence of the state $|\psi\rangle$ is $C(|\psi\rangle)=2|ad-bc|$. For the ground state (11), the concurrence can be easily obtained as follows,

$$C(|\psi_g\rangle)=\frac{2|Q(E_g-\cos\theta)|}{Q^2+(E_g-\cos\theta)^2}=|\sin\theta'|. \tag{13}$$

We can see that when spectral parameter $\theta=\theta_c=\pi/2$, $E_g\to -|Q|$. Then the concurrence of the ground state is $C(|\psi_g(\theta_c)\rangle)=1$ (i.e. the correlation between subsystems is strong at the quantum critical point). We plot the concurrence of the ground state (Please see Fig. 2).

We can say in this Yang-Baxter system, the QE is a good indicator to QPT-Like.

In this Yang-Baxter system, there is a parameter ϕ, which is the flux that can be dependent on time t in physics. If we fix parameters θ and q, and slowly vary the parameter ϕ. Then we can obtain a GP related to the ground state (11) as follows,

$$\begin{aligned}\gamma_g &= i\int_0^{2\pi}\left\langle\psi_g\left|\frac{\partial}{\partial\phi}\right|\psi_g\right\rangle d\phi\\ &=\frac{\Omega}{2},\end{aligned} \tag{14}$$

where $\Omega=2\pi(1-\cos\theta')$. When $Q=0$ the geometric phase behaves as a step function

$$\gamma_g=\begin{cases}0, & \text{for } Q=0 \quad \text{and} \quad \theta\in(0,\frac{\pi}{2}),\\ 2\pi, & \text{for } Q=0 \quad \text{and} \quad \theta\in(\frac{\pi}{2},\pi).\end{cases} \tag{15}$$

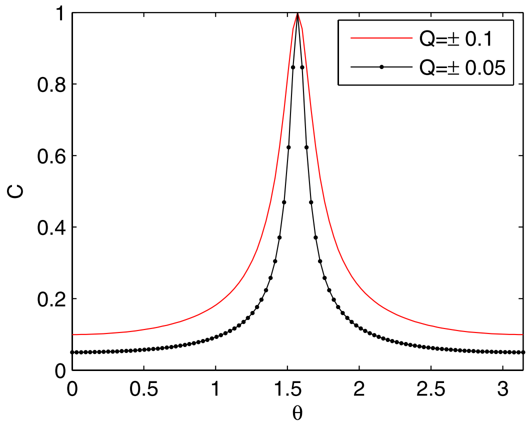

Fig. 2 Entanglement degree concurrence C as a function of θ, where $\theta \in (0, \pi)$

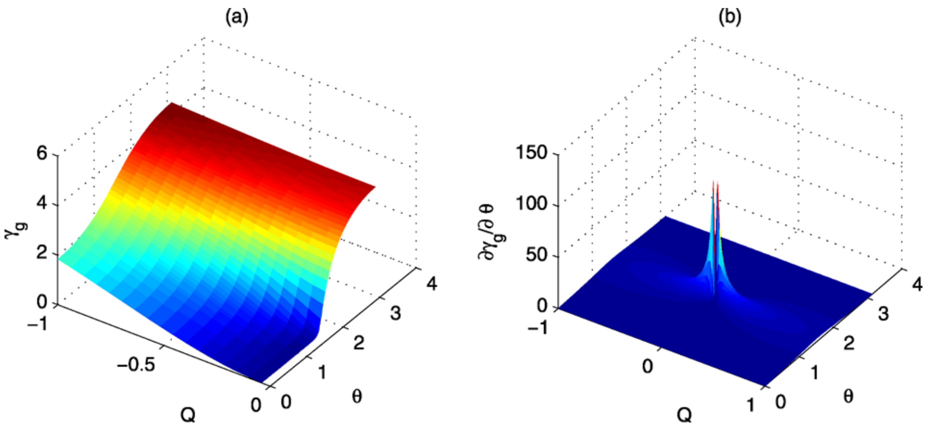

Fig. 3 (**a**) The ground state GP γ_g as a function of θ and Q, where $\theta \in (0, \pi)$ and $Q \in (-1, 0)$; (**b**) $\partial\gamma_g/\partial\theta$ versus θ and Q, where $\theta \in (0, \pi)$ and $Q \in (-1, 1)$

From above equation, we can obtain $\partial\gamma_g/\partial\theta$ as a function of θ,

$$\frac{\partial\gamma_g}{\partial\theta} = -\pi \frac{Q^2 \sin\theta}{(E_g)^3}. \tag{16}$$

The GP of the ground state γ_g and its derivative $\partial\gamma_g/\partial\theta$ versus θ and Q are plotted in Fig. 3. As can be observed, the GP behaves as the step-function near the point $Q = 0$, and at $Q = 0$, one has the discontinuity of the GP with gap of 2π.

If we fix the parameter Q, the GP of the ground state γ_g and its derivative $\partial\gamma_g/\partial\theta$ versus θ are plotted in Fig. 4.

4 Summary

In this work, by means of Yang-Baxter approach, we obtain a "*q-deformed*" two-qubit interaction Hamiltonian. In the limit $Q \to 0$ (i.e. $q \to 1$ or $d \to 2$) the crossover becomes

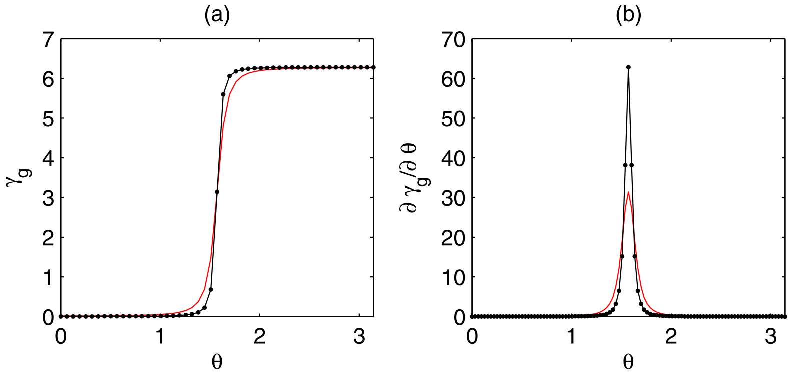

Fig. 4 (**a**) The ground state Berry phase γ_g as a function of θ; (**b**) $\partial\beta_g/\partial\theta$ as a function of θ. The $\theta \in (0, \pi)$

the QPT-Like. The Quantum entanglement and the geometric phase are good indicators. The ground state entanglement achieved maximal value 1. The first derivative of geometric phase with respect to spectral parameter θ has a non-analysis point when θ approaches $\theta_c = \pi/2$.

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References

1. Sachdev, S. (ed.): Quantum Phase Transition, 1st edn. Cambridge Univ. Press, Cambridge (1999)
2. Tian, G.S., Lin, H.Q.: Phys. Rev. B **67**, 245105 (2003)
3. Gu, S.-J.: arXiv:0811.3127 (2008)
4. Berry, M.V.: Proc. R. Soc. Lond. A **392**, 45 (1984)
5. Tian, L.J., Jin, Y.L., Jiang, Y.: Phys. Lett. B **686**, 207 (2010)
6. Osterloh, A., Amico, L., Falci, G., Fazio, R.: Nature **416**, 608 (2002)
7. Osborne, T.J., Nielsen, M.A.: Phys. Rev. A **66**, 032110 (2002)
8. Vidal, G., Latorre, J.I., Rico, E., Kitaev, A.: Phys. Rev. Lett. **90**, 227902 (2003)
9. Su, S.Q., Song, J.L., Gu, S.J.: Phys. Rev. A **74**, 032308 (2006)
10. Chen, H.D.: J. Phys. A, Math. Theor. **40**, 10215 (2007)
11. Gu, S.J., Lin, H.Q., Li, Y.Q.: Phys. Rev. A **68**, 042330 (2003)
12. Carollo, A.C.M., Pachos, J.K.: Phys. Rev. Lett. **95**, 157203 (2005)
13. Pachos, J.K., Carollo, A.C.M.: quant-ph/0602154
14. Zhu, S.L.: Phys. Rev. Lett. **96**, 077206 (2006)
15. Nesterov, A.I., Ovchinnikov, S.G.: arXiv:0907.1310 [cond-mat]
16. Peng, X., Du, J., Suter, D.: Phys. Rev. A **71**, 012307 (2005)
17. Yang, C.N.: Phys. Rev. Lett. **19**, 1312 (1967)
18. Yang, C.N.: Phys. Rev. **168**, 1920 (1968)
19. Baxter, R.J.: Exactly Solved Models in Statistical Mechanics. Academic Press, New York (1982)
20. Baxter, R.J.: Ann. Phys. **70**, 193 (1972)
21. Kauffman, L.H., Lomonaco, S.J. Jr: New J. Phys. **6**, 134 (2004)
22. Zhang, Y., Ge, M.L.: Quantum Inf. Process. **6**, 363 (2007)
23. Zhang, Y., Rowell, E.C., Wu, Y.S., Wang, Z.H., Ge, M.L.: Preprint. arXiv:0706.1761 [quant-ph] (2007)
24. Chen, J.L., Xue, K., Ge, M.L.: Phys. Rev. A **76**, 042324 (2007)
25. Chen, J.L., Xue, K., Ge, M.L.: Ann. Phys. **323**, 2614 (2008)
26. Chen, J.L., Xue, K., Ge, M.L.: Preprint. arXiv:0809.2321 [quant-ph] (2008)
27. Wang, G., Xue, K., Wu, C., Liang, H., Oh, C.H.: J. Phys. A, Math. Theor. **42**, 125207 (2009)
28. Ge, M.L., Xue, K., Wu, Y.-S.: Int. J. Mod. Phys. A **6**, 3735 (1991)
29. Kauffman, L.H., Lomonaco, S.J. Jr.: New J. Phys. **4**, 73.1–73.18 (2002)
30. Wadati, M., Deguchi, T., Akutsu, Y.: Phys. Rep. **180**, 247 (1989)